

# Classification of constant solutions for associative Yang-Baxter equation on $gl(3)$

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## Abstract

We find all non-equivalent constant solutions for classical associative Yang-Baxter equation for  $gl(3)$ . New examples found in the classification yield the corresponding quadratic trace Poisson brackets, double Poisson structures on free associative algebra with three generators and anti-Frobenius associative algebras.

MSC numbers: 17B80, 17B63, 32L81, 14H70

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# 1 Introduction

Consider skew-symmetric solutions of the associative Yang-Baxter equation (another name is the Rota-Baxter equation) [1, 2]

$$r^{23}r^{12} + r^{31}r^{23} + r^{12}r^{31} = 0, \quad r^{12} = -r^{21}. \quad (1.1)$$

Here  $r$  is a linear operator on  $V \otimes V$ , where  $V$  is  $n$ -dimensional vector space, all operators in (1.1) act in  $V \otimes V \otimes V$ , and  $r^{ij}$  means the operator  $r$  acting in the product of the  $i$ -th and  $j$ -th components. Let  $e_\alpha$ ,  $\alpha = 1, \dots, m$  be a basis in  $V$ . If

$$r(e_\alpha \otimes e_\beta) = r_{\alpha\beta}^{\sigma\epsilon} e_\sigma \otimes e_\epsilon,$$

then (1.1) is equivalent to the following system of algebraic equations:

$$r_{\alpha\beta}^{\sigma\epsilon} = -r_{\beta\alpha}^{\epsilon\sigma}. \quad (1.2)$$

and

$$r_{\alpha\beta}^{\lambda\sigma} r_{\sigma\tau}^{\mu\nu} + r_{\beta\tau}^{\mu\sigma} r_{\sigma\alpha}^{\nu\lambda} + r_{\tau\alpha}^{\nu\sigma} r_{\sigma\beta}^{\lambda\mu} = 0. \quad (1.3)$$

The coefficients of the tensor  $r$  are transformed under a change of the basis  $e_\alpha$  in the standard way:

$$r_{\alpha\beta}^{\gamma\sigma} \rightarrow g_\alpha^\lambda g_\beta^\mu h_\nu^\gamma h_\epsilon^\sigma r_{\lambda\mu}^{\nu\epsilon}, \quad (1.4)$$

where  $g_\alpha^\beta h_\beta^\gamma = \delta_\alpha^\gamma$ .

It is easy to verify that the system of algebraic equations (1.2), (1.3) is invariant with respect to the involution

$$T : \quad r_{\alpha\beta}^{\gamma\delta} \rightarrow r_{\gamma\delta}^{\alpha\beta}. \quad (1.5)$$

It is known that any solution of (1.2), (1.3) gives rise to

- the trace quadratic Poisson bracket (see [3, 4])

$$\{x_{i,\alpha}^j, x_{i',\beta}^{j'}\} = r_{\alpha\beta}^{\gamma\epsilon} x_{i,\gamma}^{j'} x_{i',\epsilon}^j, \quad (1.6)$$

where  $x_{i,\alpha}^j$  are entries of matrices  $x_\alpha$ ,  $\alpha = 1, \dots, n$ ;

- the double Poisson bracket [5, 6]

$$\{\!\{x_\alpha, x_\beta\}\!\} = r_{\alpha\beta}^{uv} x_u \otimes x_v \quad (1.7)$$

on the free associative algebra  $A = \mathbb{C} \langle x_1, \dots, x_m \rangle$ ;

- an *anti-Frobenius* associative subalgebra in  $Mat_n$  [7, 4].

All known solutions are generated by the anti-Frobenius algebras of  $n \times n$ -matrices with  $k$  zero rows (or columns), where  $k$  is any divisor of  $n$ .

In the simplest case  $n = 2$  all solutions of (1.2), (1.3) up to transformations (1.4) and the scaling  $r \rightarrow \lambda r$  were listed in [7, 6]. There exist the following two solutions:

**Case 1.**  $r_{22}^{21} = -r_{22}^{12} = 1$ .

**Case 2.**  $r_{21}^{22} = -r_{12}^{22} = 1$ .

We present here non-zero components of the tensor  $r$  only. These solutions correspond to  $2 \times 2$ -matrices with one zero row and one zero column, correspondingly.

Two solutions are called *equivalent* if they are related by transformations (1.4) or by the scaling  $r \rightarrow \lambda r$ . The main goal of this paper is a classification of all solutions (1.2), (1.3) in the case  $n = 3$  up to the equivalence.

## 2 Preliminary classification

Because of the skew-symmetry of  $r$  we have 36 unknowns and several hundred equations in system (1.2). In addition, the action (1.4) of  $gl(3)$  produces parameters in solutions. Therefore we have no chances to solve the system straightforwardly by computer algebra tools. However it turns to be possible if we use a sort of a deformation technique.

Any tensor  $r$  after transformation (1.4) with the matrix  $g = \text{diag}(1, 1, \varepsilon)$  and multiplication by  $\varepsilon^2$  has the form

$$r_{\gamma\delta}^{\alpha\beta} = a_{\gamma\delta}^{\alpha\beta} + \varepsilon b_{\gamma\delta}^{\alpha\beta} + \varepsilon^2 c_{\gamma\delta}^{\alpha\beta} + \varepsilon^3 d_{\gamma\delta}^{\alpha\beta} + \varepsilon^4 f_{\gamma\delta}^{\alpha\beta}. \quad (2.8)$$

We denote the set of all components  $a_{\gamma\delta}^{\alpha\beta}$  by  $a$  and so on. It is easy to verify that the set  $a$  may contain only one (up to the skew-symmetry) non-zero component  $a_{12}^{33}$ . The set  $b$  contains the following ten non-zero elements:

$$b_{11}^{31}, b_{21}^{31}, b_{12}^{31}, b_{22}^{31}, b_{11}^{32}, b_{21}^{32}, b_{12}^{32}, b_{22}^{32}, b_{31}^{33}, b_{32}^{33}$$

and skew-symmetric to them. The number of non-zero independent elements in  $c$ ,  $d$  and  $f$  equals 14, 10 and 1, correspondingly.

We are looking for solutions of (1.2), (1.3) in the form (2.8). It allows us to separate equations from (1.1) into several groups corresponding to different powers of  $\varepsilon$  and to solve them more or less sequentially.

In this section we find a complete list of families of solutions for (1.1). We use here mostly transformations (1.4) with  $g_3^1 = g_3^2 = g_1^3 = g_2^3 = 0$ . We call such transformations *admissible*. They act on the  $a, b, s, d, f$  in (2.8) separately i.e.  $a, b, s, d, f$  are "tensors" with respect to the group of admissible transformations. General transformations (1.4) mix together these sets.

A priori we have two classes of solutions. The first class is defined by  $a_{12}^{33} = 1$ . We call such solutions *long*. For solutions from the second class we have  $r_{12}^{33} = 0$  after any transformation

(1.4). We call them *short* solutions. It is easy to verify that the short solutions are defined by the following linear relations:

$$\begin{aligned}
r_{12}^{33} = r_{32}^{11} = r_{31}^{22} = 0, \quad r_{12}^{21} + r_{31}^{31} + r_{23}^{32} &= r_{21}^{21} + r_{13}^{31} + r_{32}^{32}, \quad r_{23}^{31} = r_{21}^{11} + r_{32}^{31}, \\
r_{31}^{32} &= r_{21}^{22} + r_{13}^{32}, \quad r_{12}^{31} = r_{21}^{31} + r_{32}^{33}, \quad r_{21}^{32} = r_{12}^{32} + r_{31}^{33}, \\
r_{32}^{21} &= r_{31}^{11} + r_{23}^{21}, \quad r_{31}^{21} = r_{13}^{21} + r_{32}^{22}.
\end{aligned} \tag{2.9}$$

**Lemma 1.** Let  $r$  be a solution of (1.1). If both  $r$  and  $T(r)$ , where  $T$  is defined by (1.5), are short, then  $r = 0$ .

**Scheme of a proof.** The invariance of the conditions  $r_{12}^{33} = r_{33}^{12} = 0$  required gives us 20 linear relations between 36 independent components of  $r$ . Finding 20 components from the linear system and substituting them into (1.2), (1.3), we arrive at a big but simple system of quadratic equations, which contains many equations of the form  $Z^2 = 0$ , where  $Z$  is a linear expression in components of  $r$ .  $\square$

In this section we consider long solutions. Note that since

$$r_{21}^{33} = -r_{12}^{33} = 1 \tag{2.10}$$

is a solution of (1.2), (1.3), the long solutions are deformations of (2.10).

Equating the coefficients at  $\varepsilon$  in (1.3) to zero, we get

$$b_{22}^{31} = b_{11}^{32} = 0, \quad b_{11}^{31} = b_{12}^{32} + b_{21}^{32}, \quad b_{22}^{32} = b_{12}^{31} + b_{21}^{31}.$$

Coefficients at  $\varepsilon^2$  allow us to express 11 of 14 non-zero elements of the set  $c$  in terms of  $b$ . Relations at  $\varepsilon^3$  give us all 10 elements of  $d$  and from  $\varepsilon^4$  we find  $f_{33}^{21} = 0$ . The elements  $b_{31}^{33}, b_{32}^{33}, b_{21}^{32}, b_{12}^{31}, b_{12}^{32}, b_{21}^{31}$  of the set  $b$  and the elements  $c_{32}^{32}, c_{32}^{31}, c_{31}^{32}$  of  $c$  remain to be undefined.

**Lemma 2.** For any long solution  $r$  the solution  $T(r)$  is short.

**Scheme of a proof.** Using the expressions for elements of  $r$  in terms of the remaining 9 elements, we can easily verify that the relations (2.9) for  $T(r)$  are fulfilled.  $\square$

It follows from Lemma 1 and Lemma 2 that the sets of long and short solutions are complementary with respect to the involution  $T$ .

Let us introduce three two-dimensional vectors

$$X = (b_{31}^{33}, b_{32}^{33}), \quad Y = (b_{21}^{32}, b_{12}^{31}), \quad Z = (b_{12}^{32}, b_{21}^{31}).$$

The remaining equations from the system (1.2), (1.3) direct onto the following separation:

**Case 1:**  $Z \neq -Y$  and **Case 2:**  $Z = -Y$ .

In Case 1 the elements  $c_{32}^{32}, c_{32}^{31}, c_{31}^{32}$  are determined and the whole system (1.2), (1.3) is equivalent to the only one relation

$$x_2 y_1 - x_1 y_2 + x_2 z_1 - x_1 z_2 + y_2 z_1 - y_1 z_2 = 0, \tag{2.11}$$

where  $x_i, y_i, z_i$  are corresponding components of  $X, Y$  and  $Z$ .

In Case 2 the system (1.2), (1.3) is equivalent to the relation

$$(x_2 y_1 - c_{32}^{32})^2 + (c_{32}^{31} + x_2 y_2)(c_{31}^{32} - x_1 y_1) = 0. \quad (2.12)$$

Now we should simplify the vectors  $X, Y, Z$  by admissible transformations. It is easy to see that the vectors are transformed as follows:

$$X \rightarrow XQ, \quad Y \rightarrow YQ, \quad Z \rightarrow ZQ,$$

where  $Q$  is arbitrary non-degenerate matrix.

In Case 1 we have the following possibilities:

- **Case 1-1:**  $X$  is not parallel to  $Y$ . In this case we reduce them to  $X = (1, 0)$ ,  $Y = (0, 1)$ . Then it follows from (2.11) that  $z_2 = z_1 - 1$ .
- **Case 1-2:**  $X$  is not parallel to  $Z$  and we arrive at  $X = (1, 0)$ ,  $Z = (0, 1)$  and  $y_2 = -y_1 - 1$ .
- **Case 1-3:**  $Y$  is not parallel to  $Z$  and we obtain  $Y = (1, 0)$ ,  $Z = (0, 1)$  and  $x_2 = x_1 + 1$ .
- **Case 1-4:** All three vectors are parallel:  $X = (u, 0)$ ,  $Y = (v, 0)$ ,  $Z = (w, 0)$ , where  $v + w \neq 0$ . Without loss of generality we put  $w = 1 - v$ .

The corresponding solutions of (1.2), (1.3) are given by:

$$\begin{aligned} r_{21}^{33} &= r_{13}^{31} = r_{12}^{33} = r_{31}^{33} = 1, \\ r_{11}^{12} &= r_{21}^{12} = r_{31}^{12} = r_{11}^{31} = r_{12}^{32} = r_{13}^{32} = r_{22}^{32} = r_{23}^{32} = r_{32}^{32} = r_{33}^{32} = u, \\ r_{12}^{11} &= r_{13}^{11} = r_{21}^{31} = r_{31}^{31} = u - 1; \end{aligned} \quad (2.13)$$

$$\begin{aligned} r_{21}^{33} &= r_{31}^{31} = r_{21}^{31} = r_{31}^{33} = 1, \\ r_{11}^{12} &= r_{21}^{21} = r_{31}^{21} = r_{22}^{23} = r_{23}^{23} = r_{32}^{23} = r_{33}^{23} = r_{11}^{31} = r_{21}^{32} = r_{31}^{32} = u, \\ r_{21}^{11} &= r_{31}^{11} = r_{21}^{13} = r_{31}^{13} = u + 1; \end{aligned} \quad (2.14)$$

$$\begin{aligned} r_{21}^{33} &= r_{13}^{13} = r_{32}^{23} = r_{11}^{31} = r_{21}^{31} = r_{21}^{32} = r_{22}^{32} = r_{21}^{33} = 1, \\ r_{11}^{12} &= r_{12}^{12} = r_{31}^{21} = r_{33}^{23} = r_{31}^{32} = r_{32}^{32} = r_{31}^{33} = u, \\ r_{21}^{12} &= r_{22}^{12} = r_{31}^{13} = r_{32}^{13} = r_{32}^{21} = r_{33}^{31} = r_{32}^{33} = u + 1; \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} r_{21}^{33} &= r_{11}^{31} = 1 & r_{31}^{33} &= r_{11}^{12} = u, & r_{21}^{32} &= v, & r_{12}^{32} &= 1 - v, & r_{31}^{32} &= uv, \\ r_{13}^{32} &= u(1 - v), & r_{12}^{22} &= v(1 - v), & r_{13}^{22} &= uv(1 - v). \end{aligned} \quad (2.16)$$

Other components of  $r$  are defined by (1.2) or are equal to zero.

In Case 2 we have the following subcases:

- **Case 2-1:**  $X$  is not parallel to  $Y$  and we reduce them to  $X = (1, 0)$ ,  $Y = (0, 1)$ . Then (2.12) can be parameterized as follows:  $c_{32}^{31} = u^2$ ,  $c_{31}^{32} = -v^2$ , and  $c_{32}^{32} = uv$ ,
- **Case 2-2:**  $X$  is parallel to  $Y$  and we have  $X = (\alpha, 0)$ ,  $Y = (\beta, 0)$ ,  $c_{32}^{31} = u^2$ ,  $c_{31}^{32} = -v^2 + \alpha\beta$ ,  $c_{32}^{32} = uv$ .

They produce the following solutions:

$$\begin{aligned}
r_{21}^{33} = r_{12}^{13} = r_{12}^{31} = r_{31}^{33} = 1, \quad r_{23}^{11} = r_{32}^{13} = r_{32}^{31} = u^2, \quad r_{12}^{12} = r_{12}^{21} = r_{32}^{23} = r_{32}^{32} = uv, \\
r_{31}^{12} = r_{31}^{21} = r_{12}^{22} = r_{32}^{22} = r_{13}^{23} = r_{13}^{32} = v^2, \\
r_{12}^{11} = u^2 - 1, \quad r_{13}^{13} = r_{13}^{31} = uv + 1, \quad r_{31}^{11} = 2uv + 1,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
r_{21}^{33} = 1, \quad r_{31}^{33} = p, \quad r_{21}^{23} = r_{21}^{32} = q, \quad r_{12}^{11} = r_{32}^{13} = r_{32}^{31} = u^2, \\
r_{12}^{12} = r_{12}^{21} = r_{13}^{13} = r_{13}^{31} = r_{32}^{32} = r_{32}^{23} = uv, \quad r_{31}^{11} = r_{32}^{12} = r_{32}^{21} = qu^2, \\
r_{31}^{23} = r_{31}^{32} = pq - v^2, \quad r_{32}^{22} = 2quv, \quad r_{31}^{22} = q(pq - v^2), \quad r_{21}^{22} = q^2 - v^2.
\end{aligned} \tag{2.18}$$

The corresponding short solutions can be found by applying the involution  $T$  to (2.13)-(2.18). This completes the preliminary classification.

### 3 Equivalence

Now we should find solutions non-equivalent with respect to the whole group of transformations (1.4). It will be shown that the parameters in all families of solutions found in Section 2 are inessential. This means that each family provides a solution corresponding to generic values of the parameters and several special solutions.

To bring a solution to a simple form we consider matrices  $M$  and  $N$  with entries  $M_j^i = r_{\alpha j}^{\alpha i}$  and  $N_j^i = r_{j\alpha}^{\alpha i}$ . Under (1.4) these matrices are transformed just as  $M \rightarrow G^{-1}MG$  and  $N \rightarrow G^{-1}NG$ . So invariants of the pair  $(M, N)$  (see [8]) are also invariants for the corresponding solution  $r$ . In particular, all coefficients of the characteristic polynomial  $S = \text{Det}(\mathbf{1} + \lambda M + \mu N)$  are invariants of  $r$ . It can be straightforwardly checked that  $S \equiv 1$  for each solution from Section 2. This implies that all eigen-values for any linear combination  $M$  and  $N$  are equal to zero.

All solutions  $r$  can be subdivided into the following three groups: Group 1 consists of solutions for which the Jordan form of  $Q = \lambda M + \mu N$  for generic  $\lambda, \mu$  is the big Jordan block;

for Group 2 the Jordan form is the small Jordan block; and  $M = N = 0$  for Group 3. For equations of Group 1 the Jordan basis for  $Q$  is a basis in which the components of  $r$  become very simple.

Consider the family of solutions (2.18) from Case 2-2. The generic solution corresponding to  $u(p + q) \neq 0$  belongs to Group 1. It is equivalent to

$$r_{12}^{11} = r_{32}^{11} = r_{13}^{12} = r_{13}^{21} = r_{23}^{22} = 1. \quad (3.19)$$

Three non-equivalent solutions of Group 2 correspond to **a**:  $q = -p, u \neq 0$ ; **b**:  $u = v = 0, p + q \neq 0$  and **c**:  $u = 0, v = \frac{p+q}{2}, p + q \neq 0$ . They are equivalent to

$$r_{32}^{11} = r_{13}^{12} = r_{13}^{21} = r_{23}^{22} = 1. \quad (3.20)$$

$$r_{23}^{22} = 1. \quad (3.21)$$

and

$$r_{13}^{12} = r_{13}^{21} = r_{23}^{22} = 1. \quad (3.22)$$

The only one non-equivalent solution of Group 3 is given by (2.10). It corresponds to  $u = v = p = q = 0$ .

It turns out that any solution of Class 2-1 is equivalent to one of (3.19)- (3.22).

Case 1-1 with  $u \neq 0, u \neq 1$  and with  $u = 1$  produces two new solutions equivalent to:

$$r_{13}^{11} = r_{13}^{12} = r_{33}^{32} = 1. \quad (3.23)$$

and

$$r_{31}^{12} = r_{33}^{32} = 1. \quad (3.24)$$

At last, Case 1-2 with  $u = -1$  gives us a solution equivalent to

$$r_{13}^{12} = r_{33}^{32} = 1. \quad (3.25)$$

### 3.1 Final result

Our previous computations lead to the following

**Theorem 1.** Any long solution  $r$  of the system (1.2), (1.3) can be reduced to one of non-equivalent equations (2.10), (3.19)- (3.25) by a transformation (1.4). Equation (3.19) belongs to Group 1, equations (3.20)- (3.25) form Group 2 and equation (2.10) belongs to Group 3. To obtain all short solutions we have to apply the involution  $T$  defined by (1.5) to the long solutions described above.  $\square$

Thus we proved that there exist 16 solutions non-equivalent with respect to the group of transformations (1.4).

Two solutions  $r$  and  $T(r)$  of Group 1, where  $r$  is defined by (3.19), correspond to anti-Frobenius associative algebras of  $3 \times 3$ -matrices with one zero row and one zero column, correspondingly. It would be interesting to investigate the anti-Frobenius associative algebras corresponding to other solutions. It could give a hint how to formulate a reasonable classification problem for anti-Frobenius associative algebras with arbitrary  $n$ . Some classification of anti-Frobenius Lie algebras can be found in [9].

**Acknowledgments.** The author is grateful to A. Odesskii, V. Roubtsov, A. Zobnin for useful discussions. The research was partially supported by the RFBR grant 11-01-00341-a.

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